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NECESSARY CONDITIONS FOR STABILITY OF DIFFEOMORPHISMS

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Abstract. S. Smale has recently given sufficient conditions for a diffeomorphism to be Ω-stable and conjectured the converse of his theorem. The purpose of this paper is to give some limited results in the direction of that converse. We prove that an Ω-stable diffeomorphism f has only hyperbolic periodic points and moreover that if p is a periodic point of period k then the kth roots of the eigenvalues of df_p^k are bounded away from the unit circle. Other results concern the necessity of transversal intersection of stable and unstable manifolds for an Ω-stable diffeomorphism.

Introduction. We will say that a diffeomorphism $f: M \to M$ of a compact manifold is Ω-*stable* if (a) there is a neighborhood $N(f)$ of f in the C^1 topology such that $g \in N(f)$ implies there is a homeomorphism h from the nonwandering set of f, $\Omega(f)$ to the nonwandering set of g, $\Omega(g)$ which satisfies $g \cdot h = h \cdot f$; and (b) if p is a periodic point of f then dim $W^s(p; f) =$ dim $W^s(h(p); g)$. Property (b) is not usually included in the definition of Ω-stable (see [3]), but it is a weak condition which is very natural and is apparently necessary for the proof of one of our lemmas (2.2). In his paper [4], S. Smale provides sufficient conditions for a diffeomorphism to be Ω-stable. One of his conditions is that the nonwandering set have a hyperbolic structure. Recall that a closed invariant set Λ is said to have a *hyperbolic structure* if

(a) There is continuous splitting of the restriction of the tangent bundle to Λ, $TM_\Lambda = E^s \oplus E^u$ which is preserved by the derivative df.

(b) There exist constants $C > 0$, $C' > 0$ and $\lambda \in (0, 1)$ and a Riemannian metric $\| \ \|$ on TM_Λ such that

$$\|df^n(v)\| < C\lambda^n \|v\| \quad \text{for } v \in E^s \text{ and } n > 0,$$

and

$$\|df^n(v)\| \geqq C'\lambda^{-n} \|v\| \quad \text{for } v \in E^u \text{ and } n > 0.$$

One would like to prove that the condition above is necessary for Ω-stability. In this paper we give results which are a start in that direction. I would like to thank

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M. Hirsch and M. Shub for valuable conversations concerning Theorem 1. Throughout we will consider only C^∞ compact manifolds and C^∞ diffeomorphisms.

THEOREM 1. *If $f: M \to M$ is Ω-stable then all the periodic points of f are hyperbolic and there exists a constant $\lambda \in (0, 1)$ such that for any periodic point p the inequalities*

$$\|df^n(v)\| \leqq C_p\lambda^n\|v\| \qquad \text{for } v \in E_p^s \text{ and } n > 0,$$
$$\|df^n(v)\| \geqq C_p^{-1}\lambda^{-n}\|v\| \quad \text{for } v \in E_p^u \text{ and } n > 0$$

are satisfied, where C_p is a positive constant depending on p.

We remark that a diffeomorphism needs to satisfy only part (a) of the definition of Ω-stability in order to satisfy the conclusion of this theorem. Part (b) of the definition is not used in the proof.

M. Shub pointed out to me that if one could obtain the above result with a single constant C independent of p then the necessity of a hyperbolic structure on the nonwandering set would follow.

We also investigate how the stable and unstable manifolds of an Ω-stable diffeomorphism fit together. Following Abraham and Smale [1] we define a *subbasic set* for $f \in \text{Diff}(M)$ to be a compact invariant set $\Lambda \subset M$ with hyperbolic structure such that $f|_\Lambda$ is topologically transitive and the periodic points are dense in Λ.

DEFINITION. Two subbasic sets Λ_1 and Λ_2 are *attached* if there are periodic points $p_1, q_1 \in \Lambda_1$ and $p_2, q_2 \in \Lambda_2$ such that $W^s(p_1) \cap W^u(p_2) \neq \varnothing$ and $W^u(q_1) \cap W^s(q_2) \neq \varnothing$.

THEOREM 2. *If $f: M \to M$ is Ω-stable and Λ_1 and Λ_2 are subbasic sets which are attached, then* $\dim E^s(\Lambda_1) = \dim E^s(\Lambda_2)$. *If $x \in \Lambda_1$ and $y \in \Lambda_2$ then $W^s(x)$ and $W^u(y)$ always intersect transversely and moreover there is a constant $\alpha > 0$ such that for any x, y and any point of intersection of $W^s(x)$ and $W^u(y)$ the angle between $W^s(x)$ and $W^u(y)$ is greater than α.*

By the angle between two subspaces V_1, V_2 in TM_x we of course mean $\inf(\text{Cos}^{-1}(v_1, v_2)_x)$ where $v_1 \in V_1$, $v_2 \in V_2$ are unit vectors and $(\ ,\ )_x$ is the inner product on TM_x.

1. The periodic points of Ω-stable diffeomorphisms.

We begin with a lemma which allows us to alter a diffeomorphism to achieve a desired derivative at a finite number of points.

(1.1) LEMMA. *Let θ be a finite set of points in M, let $Q = \bigoplus_{x\in\theta} TM_x$ and let $Q' = \bigoplus_{x\in\theta} TM_{f(x)}$. If $\varepsilon > 0$ is sufficiently small and $G: Q \to Q'$ is an isomorphism such that $\|G - df\| < \varepsilon/10$ then there exists a diffeomorphism $g: M \to M$, ε close to f in the C^1 topology, such that $dg_x = G|TM_x$ for any $x \in \theta$. Moreover if R is a compact subset of M disjoint from θ we can require $f(x) = g(x)$ for $x \in R$.*

Proof. We assume for simplicity that the Riemannian metric on M comes from an embedding $M \to R^m$ and the usual metric on R^m. Let exp be the restriction to $Q \cup Q'$ of the exponential map from TM to M determined by the metric.

We now choose a number $\delta > 0$ satisfying the following conditions:

(1) If $\theta \cup f(\theta) = \{x_i\}_{i=1}^n$ and $\hat{B}_i = \{v \mid v \in TM_{x_i}$ and $\|v\| \leqq \delta\}$ then $\exp: \hat{B}_i \to M$ is an embedding, and if $B_i = \exp(\hat{B}_i)$ then B_i and B_j are disjoint when $i \neq j$. We also make δ so small that, for all i, R is disjoint from B_i.

(2) $\|\exp(v) - v\| < \varepsilon/10$ if $v \in \hat{B}_i$ (recall $\|\ \ \|$ is the norm on R^m in which M is embedded).

(3) $\|d\exp_v\| < 1 + \varepsilon/10$ if $v \in \hat{B}_i$ and $\|d\exp_x^{-1}\| < 1 + \varepsilon/10$ if $x \in B_i$.

(4) $\hat{f}: \hat{B}_i \to Q'$ can be defined by $\hat{f}(v) = \exp^{-1}(f(\exp(v)))$ and $\|G(u) - \hat{f}(u)\| < (\varepsilon/10)\|u\|$ and $\|G(v) - d\hat{f}_u(v)\| < (\varepsilon/10)\|v\|$ hold when $u \in \hat{B}_i$ and $\hat{B}_i \subset Q$. (This is possible because $df_{x_i} = d\hat{f}_{x_i}$ and $\|G - df_{x_i}\| < \varepsilon/10$.)

(5) If $K = \sup_{x \in M} \|df_x\|$ then $\|d\exp_u - d\exp_v\| < \varepsilon/10K$ if $u, v \in \hat{B}_i$.

Choose a C^∞ real-valued function σ such that $0 \leqq \sigma(x) \leqq 1$, $\sigma(x) = 0$ if $|x| \geqq \delta$, $\sigma(x) = 1$ if $|x| \leqq \delta/4$, and $0 \leqq \sigma'(x) < 2/\delta$ for all x. Let $\rho: TM \to R$ be defined by $\rho(v) = \sigma(\|v\|)$.

We define the function $\hat{g}: \bigcup_{x_i \in \theta} \hat{B}_i \to TM$ by $\hat{g}(v) = \rho(v)G(v) + (1 - \rho(v))\hat{f}(v)$ and define $g: M \to M$ by $g(x) = \exp(\hat{g}(\exp^{-1}(x)))$ if $x \in \bigcup_{x_i \in \theta} B_i$ and $g(x) = f(x)$ otherwise. We will show that g is ε close to f in the C^1 topology.

If $x \notin \bigcup_{x_i \in \theta} B_i$, $f(x) = g(x)$ and if $x \in \bigcup_{x_i \in \theta} B_i$,

$$\begin{aligned} \|f(x) - g(x)\| &= \|\exp \cdot \hat{f} \cdot \exp^{-1}(x) - \exp \cdot \hat{g} \cdot \exp^{-1}(x)\| \\ &\leqq \varepsilon/5 + \|\hat{f} \cdot \exp^{-1}(x) - \hat{g} \cdot \exp^{-1}(x)\| \end{aligned}$$

by condition (2) above. So if $v = \exp^{-1}(x)$,

$$\begin{aligned} \|f(x) - g(x)\| &\leqq \varepsilon/5 + \|\hat{f}(v) - \rho(v)G(v) - (1 - \rho(v))\hat{f}(v)\| \\ &\leqq \varepsilon/5 + \rho(v)\|\hat{f}(v) - G(v)\| \\ &\leqq \varepsilon/5 + \varepsilon/10 < \varepsilon \quad \text{by (4) above.} \end{aligned}$$

We now check the difference in derivatives. If $v \in \bigcup_{x_i \in \theta} \hat{B}_i$,

$$d\hat{g}_v(u) = \rho(v)G(u) + d\rho_v(u)G(v) + (1 - \rho(v))\, d\hat{f}_v(u) - d\rho_v(u)\hat{f}(v).$$

So

$$\begin{aligned} \|d\hat{f}_v(u) - d\hat{g}_v(u)\| &= \|\rho(v)G(u) - \rho(v)\, d\hat{f}_v(u) + d\rho_v(u)(G(v) - \hat{f}(v))\| \\ &\leqq \rho(v)\|G(u) - d\hat{f}_v(u)\| + \|d\rho_v(u)(G(v) - \hat{f}(v))\|. \end{aligned}$$

If $\|v\| > \delta$, $\rho(v) = 0$; if $\|v\| \leqq \delta$,

$$\|G(u) - d\hat{f}_v(u)\| < (\varepsilon/10)\|u\| \quad \text{by (4) above.}$$

If $\|v\| \geqq \delta$ $d\rho_v(u) = 0$; if $\|v\| < \delta$ $\|d\rho_v\| < 2/\delta$ and $\|G(v) - \hat{f}(v)\| < (\varepsilon/10)\|v\| < \varepsilon\delta/10$, so

$$\|d\rho_v(u)(G(v) - \hat{f}(v))\| < \frac{2}{\delta}\frac{\varepsilon\delta}{10}\|u\| = \frac{\varepsilon}{5}\|u\|.$$

Hence

$$\|d\hat{f}_v(u)-d\hat{g}_v(u)\| \leqq \frac{\varepsilon}{10}\|u\|+\frac{\varepsilon}{5}\|u\| = \frac{3\varepsilon}{10}\|u\|.$$

Let $y=\exp^{-1}(x)$, $z=\hat{f}(y)$, and $w=\hat{g}(y)$. Now

$$\begin{aligned}\|df_x(v)-dg_x(v)\| &= \|d\exp_z\cdot d\hat{f}_y\cdot d\exp_x^{-1}(v)-d\exp_w\cdot d\hat{g}\cdot d\exp_x^{-1}(v)\|\\ &\leqq \|d\exp_z\cdot d\hat{f}_y\cdot d\exp_x^{-1}(v)-d\exp_w\cdot d\hat{f}_y\cdot d\exp_x^{-1}(v)\|\\ &\quad+\|d\exp_w\cdot d\hat{f}_y\cdot d\exp_x^{-1}(v)-d\exp_w\cdot d\hat{g}_y\cdot d\exp_x^{-1}(v)\|\\ &\leqq \frac{\varepsilon}{10K}\cdot K\|d\exp_x^{-1}(v)\|+\left(1+\frac{\varepsilon}{10}\right)\frac{3\varepsilon}{10}\|d\exp_x^{-1}(v)\|\end{aligned}$$

by (3) and (5) above.

So

$$\|df_x(v)-dg_x(v)\| \leqq \left[\frac{\varepsilon}{10}+\left(1+\frac{\varepsilon}{10}\right)\frac{3\varepsilon}{10}\right]\left(1+\frac{\varepsilon}{10}\right)\|v\| < \varepsilon\|v\|. \qquad \text{Q.E.D.}$$

Proof of Theorem 1. We first choose $\varepsilon>0$ sufficiently small that if g is ε close to f in the C^1 topology then f and g are Ω-conjugate.

For each periodic point p of period k we let E_p^s be the sum of the eigenspaces of df_p^k in TM_p which correspond to eigenvalues of absolute value less than or equal to one. If $V_p=\bigoplus_{i=1}^k E^s_{f^i(p)}$, then V_p is a vector space of dimension $k(\dim E_p^s)$ and we will consider the isomorphism $L_p=df\,|\,V_p$.

By the definition of E_p^s, the eigenvalues of L_p must all have absolute value less than or equal to one. If there is a $\lambda\in(0,1)$ which is an upper bound for the set of all eigenvalues of all the isomorphisms L_p (varying p), then it is clear that for each p one can choose C_p such that $\|df^n(v)\|<C_p\lambda^n\|v\|$ for $v\in E_p^s$.

On the other hand if there is no such λ then we can contradict the Ω-stability of f as follows: Choose a periodic point p with an eigenvalue γ such that $(|\gamma|^{-1}-1)<\varepsilon/20$. Let $Q_p=\bigoplus_{i=1}^k TM_{f^i(p)}$ where $k=$period of p, and define $F\colon Q_p\to Q_p$ by $F(v)=|\gamma|^{-1}\,df(v)$. F has an eigenvalue of absolute value one, namely $\gamma/|\gamma|$. If $\gamma/|\gamma|$ is not a root of unity choose $G\colon Q_p\in Q_p$ such that $\|G-F\|<\varepsilon/20$ and G has an eigenvalue which is a root of unity; otherwise let $G=F$.

It follows that

$$\|G-df\| \leqq \|G-F\|+\|F-df\| < \varepsilon/10.$$

By (1.1) there exists a diffeomorphism $g\colon M\to M$ which is an ε approximation to f and with $dg_p=G|TM_p$.

G has an eigenvalue which is a root of unity so there exists an n such that G^n has 1 as an eigenvalue. Recall that $k=$the period of p. So there exists $v\in TM_p$ such that $\|v\|=1$ and $G^{nk}(v)=v$.

Let $\Delta=\max_{0\leqq i\leqq nk}\|G^i\|$ and let $I=\{tv\mid 0\leqq|t|\leqq(\delta/4)\Delta\}$ where δ is the δ of (1.1). Then for any vector $u\in I$, $\|G^i(u)\|<\delta/4$ for any i. Referring back to the constructions in (1.1) we see if $u \in I$ then $\hat{g}^{nk}(u)=G^{nk}(u)=u$. That is, every point of exp (I) is a periodic point of g of period nk or less. However, f must be Ω-conjugate to a Kupka-Smale approximation (see [3]) which has only a finite number of periodic points of period nk or less. Thus we have contradicted the assumption that g is Ω-conjugate to f. Q.E.D.

2. **The stable manifolds of Ω-stable diffeomorphisms.** We now prove a "flattening lemma" which says that we can with an arbitrarily small perturbation of an embedding change it so there will be a flat spot around a point.

(2.1) LEMMA. *Suppose W is a submanifold of R^n, $w \in W$, U is a neighborhood of w in R^n, and U' is a compact neighborhood of w contained in U. Suppose further $h\colon U \to R^n$ is an embedding such that $h(w)=0$. Let V be the tangent space to $h(W)$ at 0 and let $\pi\colon R^n \to V$ be orthogonal projection. Then given any $\varepsilon>0$, there exists a map $h'\colon U \to R^n$ such that*

(1) *There exists a neighborhood N of w in W such that $h'(x)=\pi \cdot h(x)$ for $x \in N$.*

(2) *h' is ε close to h in the C^1 topology.*

(3) *$h(x)=h'(x)$ for $x \notin U'$.*

Proof. Let $B(\delta)=\{v+e \mid v \in V, e \in V^{\perp} \text{ and } \|v\| \leqq \delta,\ \|e\| \leqq \delta\}$, and for $v \in V$ let $B_v(\delta)=\pi^{-1}(v) \cap B(\delta)$. Since $h(W)$ is tangent to V at 0 there exists $\delta_1>0$ such that, for each $v \in V$ with $\|v\|<\delta_1$, $B_v(\delta_1) \cap h(W)$ is a single point. Define $u\colon B(\delta_1) \to R^n$ by $u(x)=B_{\pi(x)}(\delta_1) \cap h(W)$. u is simply projection on $h(W)$ along the planes orthogonal to V. Since the manifolds $h(W)$ and V are tangent at 0, the functions π and u are tangent at 0, i.e.

$$\lim_{\|x\| \to 0} \frac{\|\pi(x)-u(x)\|}{\|x\|} = 0.$$

Hence we can choose $\delta_2<\delta_1$ such that

(1) $\delta_2<\varepsilon$ and $h^{-1}(B(\delta_2)) \subset U'$.

(2) $\|\pi(x)-u(x)\|<(\varepsilon/4K)\|x\|$ when $\|x\|<\delta_2$ where $K=\sup \|dh_z\|$ over $z \in U'$.

(3) $\|d\pi_x-du_x\|<\varepsilon/2K$ when $\|x\|<\delta_2$.

We also choose a C^∞ function $\sigma\colon R \to R$ such that $0 \leqq \sigma(t) \leqq 1$, $\sigma(t)=0$ when $|t|>\delta_2$, $\sigma(t)=1$ when $|t|<\delta_2/4$, and $|d\sigma/dt|<2/\delta_2$ for all t. Let $\rho(x)=\sigma(\|x\|)$.

Now define $h'\colon U \to R^n$ by $h'(z)=h(z)$ if $z \notin h^{-1}(B(\delta_2))$ and $h'(z)=h(z)+\rho(h(z))(\pi(h(z))-u(h(z)))$ otherwise.

If N is any neighborhood of w in W such that $N \subset h^{-1}(B(\delta_2/4))$ then for $z \in N$,

$$\begin{aligned} h'(z) &= h(z)+\rho(h(z))(\pi(h(z))-u(h(z))) \\ &= h(z)+\pi(h(z))-h(z) = \pi(h(z)). \end{aligned}$$

Also if $h(z) \in B(\delta_2)$,

$$\|h(z)-h'(z)\| \leqq \rho(h(z))\|\pi(h(z))-u(h(z))\| < \delta_2 < \varepsilon,$$

and

$$\begin{aligned} \|dh'_z-dh_z\| &\leqq \rho(h(z))\|d\pi_{h(z)}\cdot dh_z-du_{h(z)}\cdot dh_z\|+\|d\rho_{h(z)}\cdot dh_z\|\cdot\|\pi(h(z))-u(h(z))\| \\ &\leqq K(\varepsilon/2K)+(2/\delta_2)K(\varepsilon/4K)\|h(z)\| < \varepsilon/2+\varepsilon/2 = \varepsilon. \end{aligned}$$

Q.E.D.

(2.2) LEMMA. *If $f\colon M \to M$ is Ω-stable and p and q are periodic points with* orb (p) *and* orb (q) *attached then* $\dim W^s(p) = \dim W^s(q)$.

Proof. We can assume without loss of generality that $W^s(p) \cap W^u(q) \neq \varnothing$ and $W^u(p) \cap W^s(q') \neq \varnothing$ where $q' \in \operatorname{orb}(q)$. Suppose $\dim W^s(p) < \dim W^s(q)$ and let $x \in W^s(p) \cap W^u(q)$. We can perturb f slightly to f_1, leaving it fixed on a neighborhood of $\operatorname{orb}(x) = \{f^n(x) \mid -\infty < n < \infty\}$, so that $W^s(q'; f_1)$ and $W^u(p; f_1)$ will have a point of transversal intersection. By making the perturbation f_1 close enough to f we can guarantee that f_1 is Ω-stable. We now have $x \in W^s(p, f_1) \cap W^u(q, f_1)$ and $W^s(q'; f_1)$ and $W^u(p; f_1)$ have a point of transversal intersection. From this it follows easily that $x \in \Omega(f_1)$ (see [1]). We now choose a Kupka-Smale approximation (see [3] for definition) to f_1, f_2 which is sufficiently close to f_1 that there is an Ω-conjugacy h' between f_1 and f_2 and that $\dim W^s(h'(p)) = \dim W^s(p)$ and $\dim W^u(h'(q)) = \dim W^u(q)$. Since $\dim W^s(h'(p)) + \dim W^u(h'(q)) < \dim M$ and these two manifolds have transverse intersection (f_2 is Kupka-Smale) we have $W^s(\operatorname{orb}(h'(p))) \cap W^u(\operatorname{orb}(h'(q))) = \varnothing$.

Thus f_1 and f_2 cannot possibly be Ω-conjugate, since any conjugacy would have to carry x to a point in $W^s(h'(p)) \cap W^u(h'(q))$. We have contradicted the assumption that $\dim W^s(p) < \dim W^s(q)$. A similar argument contradicts the reverse inequality and we conclude $\dim W^s(p) = \dim W^s(q)$. Q.E.D.

(2.3) LEMMA. *If $f\colon M \to M$ is Ω-stable and p and q are periodic points with* orb (p) *attached to* orb (q) *then* $W^s(\operatorname{orb}(p))$ and $W^u(\operatorname{orb}(q))$ *always intersect transversely.*

Proof. Suppose to the contrary that x is a point of nontransversal intersection of $W^s(p)$ and $W^u(q)$. By (2.2), $\dim W^u(p) + \dim W^s(q) = n$ so we can perturb f to f_1 so that f_1 is Ω-stable, so $f = f_1$ on a neighborhood of $\operatorname{orb}(x) = \{f^n(x) \mid -\infty < n < \infty\}$, and so that $W^u(p; f_1)$ and $W^s(q'; f_1)$ have a point of transversal intersection for some $q' \in \operatorname{orb}(q)$. There exists an $\varepsilon' > 0$ such that if g is ε' close to f_1 then g is Ω-conjugate to f_1 and the stable and unstable manifolds of g corresponding to $W^s(q'; f_1)$ and $W^u(p; f_1)$ respectively still have a point of transversal intersection.

x is a point of nontransversal intersection of $W^s(p; f_1)$ and $W^u(q; f_1)$. Let U be a neighborhood of x in M such that $U, f_1(U)$ and $f_1^{-1}(U)$ are disjoint and $\exp_x^{-1}$ is a diffeomorphism of U onto its image in TM_x. Moreover we require that U be sufficiently small that if $N_s =$ the component of x in $W^s(p) \cap U$ and $N_u =$ the component of x in $W^u(q) \cap U$ then $\bigcup_{i=1}^{\infty} f_1^i(N_s)$ and $\bigcup_{i=1}^{\infty} f_1^{-i}(N_u)$ are disjoint from U.

Let $V^s = TW^s(p)_x$ and let $V^u = TW^u(q)_x$, then there exists a nonzero vector $v \in V^s \cap V^u$. Choose $\delta > 0$ such that $\exp_x(tv) \in U$ for $-\delta \leqq t \leqq \delta$ and let $J = \exp_x(\{tv \mid -\delta \leqq t \leqq \delta\})$.

Choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$ and if g is ε close to f_1 then g is Ω-conjugate to f_1. Let $k\colon f_1^{-1}(U) \to TM_x$ be given by $k = \exp_x^{-1} \cdot f_1$. Choose $\varepsilon_1 > 0$ so that if $k'\colon f_1^{-1}(U) \to TM_x$ is ε_1 close to k then $\exp_x \cdot k'$ is $\varepsilon/4$ close to f_1. Now by (2.1) we can find

$k_1: f_1^{-1}(U) \to TM_x$ such that (1) $k_1 = k$ outside some neighborhood of $f_1^{-1}(x)$ which is contained in $f_1^{-1}(U)$, (2) k_1 is ε_1 close to k, and (3) $k_1(W^u(q) \cap f_1^{-1}(U))$ contains a neighborhood of 0 in V^u. Thus if we define $f_2: M \to M$ by $f_2(x) = f_1(x)$ for $x \notin f_1^{-1}(U)$ and $f_2(x) = \exp_x \cdot k_1$ for $x \in f_1^{-1}(U)$ then f_2 is $\varepsilon/4$ close to f_1 and $W^u(q, f_2)$ contains a subinterval of J.

Similarly we can construct f_3 such that $f_3(x) = f_2(x)$ for $x \notin f_1(U)$, f_3 is $\varepsilon/4$ close to f_2 and $W^s(p, f_3)$ contains a subinterval of J. If J' is the subinterval of J contained in $W^u(q_1 f_2)$ then $f_3^{-i}(J')$ is disjoint from $f_1(U)$ so $f_3^{-i}(J') = f_2^{-i}(J')$ hence $J' \subset W^u(q; f_3)$.

Thus there is a neighborhood of x in J which is contained in $W^s(p; f_3) \cap W^u(q; f_3)$. Also since $W^u(p; f_3)$ and $W^u(q'; f_3)$ have a point of transversal intersection each point of $W^s(p; f_3) \cap W^u(q; f_3)$ is nonwandering (see the theorem of §1 in [1]).

This leads to a contradiction however since f_3 is Ω-conjugate to f_1 which is Ω-stable and hence f_3 is Ω-conjugate to a sufficiently close Kupka-Smale approximation to f_1. However, any Kupka-Smale approximation has only a countable number of points in the intersection of a stable and unstable manifold of complimentary dimensions, while $\dim W^s(p; f_3) + \dim W^u(q; f_3) = n$ and their intersection contains an interval of nonwandering points. Hence our initial assumption of the existence of a nontransversal point of intersection must have been false. Q.E.D.

Proof of Theorem 2. Choose $\varepsilon > 0$ so that if g is ε close to f in the C^1 metric then g is Ω-stable and Ω-conjugate to f. By hypothesis there are periodic points $p_1, q_1 \in \Lambda_1$ and $p_2, q_2 \in \Lambda_2$ such that $W^u(q_1) \cap W^s(q_2) \neq \varnothing$ and $W^s(p_1) \cap W^u(p_2) \neq \varnothing$. Suppose without loss of generality that $\dim W^s(p_1) + \dim W^u(p_2) \geqq n$ (otherwise consider q_1 and q_2). Let $w \in W^u(q_1) \cap W^s(q_2)$ then we can obtain an approximation f_1 which is $\varepsilon/4$ close to f in the C^1 metric, which is equal to f on the closure of $\operatorname{orb}(w) \cup \Lambda_1 \cup \Lambda_2$, and which has the property that $W^s(p_1; f_1)$ and $W^u(p_2; f_1)$ have a point of transversal intersection.

The argument of (7.6) (a) in Smale's paper [3] shows that $W^s(\operatorname{orb}(q_1); f_1)$ is dense in Λ_1 and $W^u(\operatorname{orb}(q_2); f_1)$ is dense in Λ_2. Hence $W^s(\operatorname{orb}(q_1); f_1)$ is dense in $W^s(\Lambda_1; f_1)$ and $W^u(\operatorname{orb}(q_2); f_1)$ is dense in $W^u(\Lambda_2; f_1)$. By the continuity of local stable manifolds (see 3.2 of [2]) there must exist a point of intersection of $W^s(\operatorname{orb}(q_1); f_1)$ and $W^u(\operatorname{orb}(q_2); f_1)$ since $W^s(p_1; f_1)$ and $W^u(p_2; f_1)$ intersect transversely. That is, considering the diffeomorphism f_1, $\operatorname{orb}(q_1)$ and $\operatorname{orb}(q_2)$ are attached. Hence by (2.2), $\dim W^s(q_1) = \dim W^s(q_2)$. Thus $\dim E^s(\Lambda_1) = \dim E^s(\Lambda_2)$.

We now assume that there is no lower bound on the angles of intersection of $W^s(\Lambda_1)$ and $W^u(\Lambda_2)$ and derive a contradiction to Ω-stability of f.

We first choose $\beta > 0$ such that

(1) If we have unit vectors $v_1, v_2 \in TM_x$ with the angle between v_1 and v_2 less than β then there exists $L: TM_x \to TM_x$ such that $L(v_1) = v_2$ and

$$\|L - \mathrm{id}\| < (\varepsilon/20)(\sup \|df_x\| + \varepsilon)^{-1}.$$

(2) For any $x \in \Lambda_1$ or Λ_2 the angle between E_x^s and E_x^u is greater than β.

Suppose there are points $x_1 \in \Lambda_1$, $x_2 \in \Lambda_2$ and z such that $W^s(x_1; f_1)$ and $W^u(x_2; f_1)$ intersect at z with the angle between them less than $\beta/2$ (this implies $z \notin \Lambda_1 \cup \Lambda_2$). If $W^s(x_1; f_1)$ does not intersect $W^u(x_2; f_1)$ transversely then we approximate f_1 by f_2 so that f_2 is $\varepsilon/4$ close to f_1 in the C^1 metric, f_2 equals f_1 on a neighborhood of $\Lambda_1 \cup \Lambda_2 \cup \operatorname{orb}(w)$ and $W^s(x_1; f_2)$ intersects $W^u(x_2; f_2)$ transversely at z but with the angle between them less than β.

As before $W^s(\operatorname{orb}(q_1); f_2)$ is dense in Λ_1 and $W^u(\operatorname{orb}(q_2); f_2)$ is dense in Λ_2. Hence we can conclude from the continuity of the local stable manifolds (3.2 of [2]) that there is a point z' near z where $W^s(\operatorname{orb}(q_1); f_2)$ intersects $W^u(\operatorname{orb}(q_2); f_2)$ and the angle between them is less than β. Hence there are unit vectors $v^s \in TM_{z'}$ tangent to $W^s(\operatorname{orb}(q_1); f_2)$ and $v^u \in TM_{z'}$ tangent to $W^u(\operatorname{orb}(q_2); f_2)$ and there is an isomorphism $L\colon TM_{z'} \to TM_{z'}$ such that $L(v^u) = v^s$ and $\|L \cdot df_2 - df_2\| < \varepsilon/20$.

Let $y' = f_2^{-1}(z')$ and define $G\colon TM_{y'} \to TM_{z'}$ by $G = L \cdot df_2$. Let $Q = \{f_2^n(y') \mid n \neq 0\} \cup \operatorname{orb}(w)$. We now apply (1.1) to the diffeomorphism f_2 with $\theta = y'$, G as we have constructed it and R a compact neighborhood $\Lambda_1 \cup \Lambda_2 \cup Q$ which is disjoint from y'.

We obtain a diffeomorphism $g\colon M \to M$ which is Ω-stable and ε-close to f in the C^1 metric. Since $g = f_2$ on R we have $dg^n(v^s) = df_2^n(v^s)$ for $n \geqq 0$. On the other hand $dg^{-n}(v^s) = df_2^{-n}(v^u)$ for $n > 0$. So v^s is tangent to both $W^s(\operatorname{orb}(q_1); g)$ and $W^u(\operatorname{orb}(q_2); g)$. Hence by (2.3) g cannot be Ω-stable which is a contradiction. Q.E.D.

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